Wave turbulence and cascade in chiral fluid

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Chiral magnetic effect (CME)

\[ \vec{j}_{\text{CME}} = C_A \mu_A \vec{B} \]

- CME: simple equation, rich physics.
- Interesting implications for dynamics of chiral system.
Applications of chiral dynamics at varieties of scales

Primordial Magnetic Fields: \( \sim 10^{16} \) m (light year)

Neutron Star: \( \sim 10^3 \) m.

Dirac Semimetals: \( \sim 10^{-2} \) m

Hot QCD matter at RHIC: \( \sim 10^{-15} \) m (fermi)
Example: chiral plasma instability

- Changing of magnetic helicity induces chiral charge imbalance and CME current.

\[ \partial_\mu J_A^\mu = C_A \vec{E} \cdot \vec{B} \]

- CME current in turn affects configuration of magnetic field.

\[ \nabla \times \vec{B} = C_A \mu_A \vec{B} + \ldots \]

- Chiral plasma instability: helical magnetic field with \( k/\sigma_A = C_A \mu_A \) will grow exponentially.
Chiral Plasma Instability: Energy talks.

\[ k > \sigma_A \]

\[ k < \sigma_A \]
Chiral Fermions play the role of “chiral” battery.

Self-similarly inverse cascade driven by chiral plasma instability.

(Evolution of Maxwell-Chern-Simons equation, Y. Hirono, D. Kharzeev and YY, PRD 15’)

(see also M.Joyce and M.Shaposhnikov, 1997; A.Boyarsky, J.Frohlich, O.Ruchayskiy … and K. Tuchin and X. Xia’s talk)
Solving Maxwell-Chern-Simon theory in FRW background.
Outline

• A step forward: inverse cascade in anomalous Magneto-hydrodynamics (aMHD).

  Y. Hirono, D. Kharzeev, D. Teaney, Ho-Ung Yee and YY, in progress

• Chiral Magnetic Wave turbulence (briefly).

  Yacine Mehtar-Tani (INT) and YY, in progress
Hydrodynamics and quantum anomaly

• Navier-Stoke equation: simple equation, rich physics.

  “Often, people in some unjustified fear of physics say you can’t write an equation for life. Well, perhaps we can.” —Feynman

• MHD was developed in 1930s by combing Maxwell’s theory and Navier-Stokes equations.

• Very recently (2007-now): quantum effects are implemented in hydrodynamics of chiral fluids:

  \[ J^\mu = \sigma E^\mu + \sigma_A B^\mu \]  
  (Son-Surowka, 2009; D. Kharzeev-Yee, 2011; Giovanni, 2016)

• Anomalous MHD is still developing (progress made in the “near perfect” aMHD, K. Hattori, Y. Hirono, Ho-Ung Yee and YY, in preparation, see also Kovtun’s talk).
Simplified aMHD equations

\[
[\partial_t + \mathbf{v} \cdot \nabla] \mathbf{v} = \frac{1}{\omega} (\mathbf{B} \cdot \nabla) \mathbf{B} + \nu_\eta \nabla^2 \mathbf{v} \quad \text{ (Acceleration = force)}
\]

\[
[\partial_t + \mathbf{v} \cdot \nabla] \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} + \frac{1}{\sigma} \nabla^2 \mathbf{B} + \frac{\sigma_A}{\sigma} (\nabla \times \mathbf{B}) \quad \text{ (Faraday’s law )}
\]

\[
[\partial_t + \mathbf{v} \cdot \nabla] n_A = \frac{C_A}{\sigma} \left[ (\nabla \times \mathbf{B}) \cdot \mathbf{B} - \sigma_A \mathbf{B}^2 \right] . \quad \text{ (Anomaly )}
\]

(In “non-relativistic” and incompressible limit with isotropic transport coefficient)

\[
\frac{T}{\sigma} \ll 1
\]

- Without anomaly equation: conventional MHD.
- Without velocity field: Maxwell-Chern-Simons equations.
- A convenient decomposition: \( \tilde{\mathbf{B}}(t, \tilde{x}) = \tilde{\mathbf{B}}_0 + \sqrt{\epsilon + p} \tilde{b} \)
Feature of aMHD I: instability

- Alfven wave: eigenstate of linearized aMHD.

\[ l^R_+ = +i u_A k \]
\[ l^R_- = -i u_A k \]
R-helical Alfven wave moving along \( \hat{B} \)
R-helical Alfven wave moving oppo. to \( \hat{B} \)

- \( \sigma_A \) breaks the degeneracy in the dispersion of R/L Alfven and induce aMHD instability (K. Hattori, Y. Hirono, Ho-Ung Yee and YY, in preparation).

Unstable mode:

\[ k < k_c = \frac{\sigma_A}{1 + \sigma(\frac{n}{w})} \]

**aMHD instability:** helicity transfer between Helical Alfven wave and axial charge density.
Feature of aMHD II: $B_0$ relaxes axial charge

- Zero current configuration at finite $\sigma_A$:

$$\vec{j}_V = \sigma_A \vec{B}_0 + \sigma \vec{E} = 0 \rightarrow \vec{E} = -\frac{\sigma_A \vec{B}}{\sigma} \rightarrow \partial_t n_A = C_A \vec{E} \cdot \vec{B} = -\frac{n_A}{\tau_A}$$

- Anomaly equation can be written as:

$$\tau_A = \frac{\sigma \chi_A}{C_A^2 B_0^2}$$

$$\partial_t \left( n_A + \underbrace{C_A h_m^b}_{\text{inhomogeneous part}} \right) = -\frac{n_A}{\tau_A}$$

- $\tau_A$ determines how fast those helical modes with finite $k$ are flowing to zero mode (condensation).
Full dynamical evolution

• Solving coupled evolution equation of aMHD.

• No-transverse moment => no advection term which induces conventional Alfven wave turbulence. (future: interplay between conventional Alfven wave turbulence and inverse cascade)

• Other technical simplification: homogeneous but time-dependent axial charge, right helical waves only.

(Y. Hirono, D. Kharzeev, D. Teaney, Ho-Ung Yee and YY, in progress)
Results: spectrum functions

- Inverse cascade for both helical magnetic field and helical velocity field.
Results: inverse cascade

- Two steps:
  - Step 1: $k_b \sim k_v$ and $E_v = E_b$ (Alfven effect);
  - Step 2: $k_b(t) \sim \sigma_A(t)$ (anomaly).

- $E_v$, $E_b$, and $h_v$ vanishes at different rate during inverse cascade.

\[
E_v = \frac{1}{2} (\epsilon + p) v^2, \\
E_B = \frac{1}{2} (\epsilon + p) b^2 \\
h_f \sim \mathbf{v} \cdot \mathbf{w} \sim k_v E_v, \\
h_m = \mathbf{A} \cdot \mathbf{B} \sim k_B^{-1} E_B.
\]
Alfven effect

- Alfven effect: constructive interference of Alfven wave for the combination $v+b$ or $v - b$, i.e., $\langle E_v \rangle = \langle E_b \rangle$ (NB: observed in solar wind!).

(solar wind data on February 21, 1978 observed by Voyager 2, 1606.03308)
Alfven effect and its Implications

- Helical fluid velocity would induce the helical magnetic (or vice versa)
• Any late stage self-similar evolution for aMHD?

• Naive expectation: exponential decay of magnetic helicity in long time limit.

\[
\partial_t \left( n_A + \underbrace{C_A h_m^b}_{\text{inhomogeneous part}} \right) = -\frac{n_A}{\tau_A}
\]
Two scaling regime

- Keeping a large but finite axial relaxation time $\tau_A$: two scaling regimes emerge!
Self-similar evolution

- Two different sets of scaling exponents!

\[ g(t^{\frac{1}{2}} k) = t^{-\alpha} \tilde{g}(t^{\beta} k), \quad \sigma_A(t) \sim t^{-\gamma} \]
Effects of finite relaxation time

- Implication: finite relaxation time does not imply exponential decay. For sufficient large $\tau_A$, it will be power law decay. (similar to hydrodynamical tail).

\[
\partial_t \left( n_A + \underbrace{C_A h_m^b}_{\text{inhomogeneous part}} \right) = -\frac{n_A}{\tau_A}
\]
Alfven wave has a long life time when conductivity is large.

Next: opposite situation, small conductivity or diffusive constant $D \sim \sigma$. Chiral magnetic wave has a longer life time.

Chiral magnetic wave turbulence
Wave turbulence

- Turbulence: ubiquitous in physics.

- (weak) wave turbulence (Kolmogrov-Zakharov): turbulence driven by an ensemble of weakly interacting waves (similar to mode-mode coupling theory for hydro. fluctuations).

- Universality: scaling exponents only depend on dispersion of the wave, spatial dimension and number of waves involved in interaction.

(Falkovich et al, physics today)
Chiral magnetic wave (CMW)

\[ \partial_t n_R = -C_A (\vec{B} \cdot \nabla) \left[ T^{-1} \frac{\partial s(T, n_R)}{\partial n_R} \right] + \text{Diffusion} \]

- Linearized equation: CMW.

\[ \omega(\vec{k}) = -k \vec{v}_{CMW} \]

\[ \vec{v}_{CMW} = \frac{C_A \vec{B}}{\chi(T)} \]

- Nonlinearity from E.o.S:

\[ s(T, n_R) = \frac{1}{2\chi(T)} n_R^2 + \frac{a(T)}{4} n_R^4 + \ldots \]

- Chiral fluid: perhaps the simplest set-up to realize wave turbulence.
  - CMW is directional (along B)
  - Three-wave interaction is not allowed.
Turbulence and Direct cascade of chiral charge spectrum

- Spectrum functions:
  \[ \langle n_R(t, \vec{k}) n_R(t, \vec{k}') \rangle \sim g^R(t, \vec{k}) \delta^3(\vec{k}' + \vec{k}), \]

- A kinetic equation for spectrum functions (following Book by E. Zakharov et al, 1992)
  \[ \partial_t g^R(t, \vec{k}) = - \underbrace{\mathcal{I}[g^R]}_{\text{Collisions of 4-CMW}} \]

- Static solution:
  \[ g^R(k_z, k_\perp) \sim k_z^{-a} k_\perp^{-b} \]
Preliminary results

- Neglecting transverse dynamics:

\[ g^R(k_z) \sim k_z^{-5/3} \text{ or } k_z^{-4/3} \]

(Yacine Mehtar-Tani and YY, in progress)

Pumping axial charge at large scale

Energy transfer due to interactions among CMW

Cascade terminated by diffusion at short scale

- CMW turbulence scaling can be tested by stochastic anomalous hydro simulation (or real time lattice simulation).

- Weyl Semi-metal?
Summary and outlook
Summary

- aMHD evolution: Alfvén wave transfers helicity/energy between magnetic field and fluid.
  - Potential contribution to vortical effect and polarization.
  - Hydro. fluctuations might generate magnetic field.

- Long time tail for axial charge and helical magnetic field: helical magnetic field and axial charge might live longer than expected.

- Possible wave turbulence in Chiral fluid due to Chiral magnetic wave.

- Many interesting physics ahead.
Back-up slides
“Near perfect” MHD

• Magnetic diffusive rate: \( \Gamma_B \sim \frac{k^2}{\sigma} \) (v.s. \( \Gamma_{\text{sound}} \sim \left( \frac{\eta}{sT} \right) k^2 \))

• “Near perfect” MHD: \( \frac{T}{\sigma} \ll 1 \)

• Dynamical variables in the “near perfect” aMHD: energy density, flow, magnetic field and axial charge.

• Expressing electric field and EM current by gradient expansion (controlled by \( k/\sigma \)), e.g.,

\[
\begin{align*}
\vec{E} + \vec{v} \times \vec{B} &= \sigma^{-1} \left( \nabla \times \vec{B} - \sigma_A \vec{B} \right) \\
&= \sigma^{-1} \left( E^\mu (\nabla \times \vec{B})_\mu - \sigma_A \vec{B} \right)
\end{align*}
\]

• The CME can be derived from second law of thermodynamics! (K. Hattori, Y. Hirono, Ho-Ung Yee and YY, in preparation)
• Chirality can be transferred among various forms via anomaly.

• The dynamical evolution induced by anomaly exhibiting possible unique and universal pattern.

• Anomalous magneto-hydrodynamics: hydrodynamics in 21st century!
QCD: simply Lagrangian, rich phenomenologies.

QCD matter: intriguing and interesting (more is different)!

Chiral anomaly is important for (microscopic) particle physics, e.g. $\pi^0$ decay to two photons.

**Macroscopic manifestation of quantum anomaly?**
• Relevant spectrum functions:
\[ \langle b^R(t, \vec{k}) b^R(t, \vec{k}') \rangle \sim g_{bb}^R(t, \vec{k}) \delta^3(\vec{k}' + \vec{k}) \]

Similar definition for:
\[ g_{vv}^{R,L}(t, \vec{k}), g_{vb}^{R,L}(t, \vec{k}) \]

• Moments of spectrum function are related to helictites and energy density.
\[ \langle E_v \rangle \equiv \langle v^2(t, \vec{x}) \rangle = \int \overline{k} \left( g_{vv}^R(t, \vec{k}) + g_{vv}^L(t, \vec{k}) \right) \]
\[ \langle h_v \rangle \propto \langle \vec{v}(t, \vec{x}) \cdot \vec{\omega}(t, \vec{x}) \rangle = \int \overline{k} \left( g_{vv}^R(t, \vec{k}) - g_{vv}^L(t, \vec{k}) \right) \]

Similar relation for: