Hydrodynamics and Anomalies

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Plan of the talk

- Hydrodynamics
- Anomalies
- Gauge/gravity duality
- Hydrodynamics with anomalies
- Anomalies in kinetic theory (also Misha Stephanov’s talk)
Relativistic hydrodynamics

• The conservation of energy, momentum, baryon charge

\[ \nabla_\mu T^{\mu\nu} = 0 \quad \nabla_\mu j^\mu = 0 \]

“Constitutive relations”

\[ \vec{j} = \rho \vec{v} - D \vec{\nabla} \rho \]

advection diffusion

a similar equation for \[ T^{\mu\nu} \]
Relativistic hydrodynamics has many applications in astrophysics and heavy ion collisions.
Quantum phenomenon in fluid dynamics

- Hydrodynamics is considered a classical theory
- Very recently (2007-now): quantum effects discovered in hydrodynamics of normal fluids
  - related to triangle anomalies
- It was almost an accidental discovery, made through gauge-gravity duality
Triangle diagram

\[ \pi^0 \rightarrow \gamma \gamma \]
Chirality in gauge theories

- Massless fermions: can be of either chiralities

Chirality does not change in when the particle interacts with gauge fields

But chirality is not conserved in quantum theory: anomalies
Landau levels

- To understand anomalies, we start with quantum mechanics of a massless fermion in a magnetic field
Massless fermion in a magnetic field

\[ E^2 = p_z^2 + 2nB \]
Anomalies

Turn on electric field for some duration of time
Anomalies

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Turn on electric field for some duration of time

\[ B \rightarrow E \]

\[ L \leftarrow \varepsilon \rightarrow R \]

\[ P_z \]
Anomalies

Turn on electric field for some duration of time

\[ \frac{d}{dt}(N_R - N_L) \sim E \cdot B \]

\[ \partial_\mu j^{5\mu} = \frac{e^2}{4\pi^2} \vec{E} \cdot \vec{B} \]
Anomalies and hydrodynamics

• That anomalies can change hydrodynamics was not expected
  • consider a gas of chiral particles
  • conservation laws valid in the absence of external gauge field

Breakthrough: a framework in which anomalies and hydrodynamics can be studied at the same time
Gauge/gravity duality ("holography")

Maldacena (1997): duality between QFT and string theory

N=4 super Yang-Mills theory

string theory in AdS$_5 \times$S$^5$ space

\[ ds^2 = \frac{r^2}{R^2} (-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \]
Duality as a tool for QFT

- Gauge/gravity duality is particularly useful in the strong coupling regime of QFT

\[ g^2N_c \ll 1: \text{string theory becomes gravity} \]

Difficult regime in field theory = easy in string theory
Hydrodynamics from holography

- Quark-gluon plasma (in N=4 SYM theory) = black hole in AdS space with flat horizons
- Hydrodynamics \sim dynamics of black holes with flat horizons
- “fluid-gravity correspondence”
Puzzle form rotating BH

Bhattacharyya, Lahiri, Loganayagam, Minwalla 2007

• Rotating black holes ~ rotating fluids
• Black hole thermodynamics = fluid thermodynamics
• Can check the correspondence explicitly:
  • black hole solutions known exactly
  • fluid dynamic solution is known in expansion over $1/(TR)$
Mismatch

- BH and fluids match perfectly to leading order in $1/(TR)$
- but while BH thermodynamics has a correction $\sim 1/(TR)$, standard fluid dynamics predicts that the first correction is $1/(TR)^2$
- The discrepancy remained unresolved for 2 years
Resolution

• In 2009 it was found that the mismatch is due to a new term in the hydrodynamic equations of N=4 super-Yang-Mills theory

\[ \vec{j} = \rho \vec{v} - D \vec{\nabla} \rho + \xi(T, \mu) \vec{\nabla} \times \vec{v} \]

Erdmenger, Haack, Kaminski, Yarom
Banerjee, Bhattacharya, Bhattacharyya, Dutta, Loganayagam, Surówka

Although the new term is not forbidden by parity (current in N=4 SYM is chiral), this term seems to be very strange...
Chiral separation by rotation?
Chiral separation by rotation?
Chiral separation by rotation?
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Chiral separation by rotation?
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Chiral separation by rotation?
A stronger objection

- There is no mention of a CVE in volume 6 of Landau and Lifshitz (Fluid Mechanics) and in any of the 10 volumes!
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The entropy argument

- The hydrodynamic equation is not time-reversal invariant
- There must exist an entropy, expressible in terms of hydrodynamic variables (local velocity, T, etc.) which does not decrease with time
- This condition allows only shear viscosity, bulk viscosity, thermal conductivity (charge diffusion coefficient)
Dissipative terms

Standard textbook manipulations (single U(1) charge)

\[ \partial_\mu [( \epsilon + P ) u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu \nu} = 0 \]

\[ \partial_\mu (nu^\mu) + \partial_\mu \nu^\mu = 0 \]
Dissipative terms

Standard textbook manipulations (single U(1) charge)

\[ \partial_\mu [(Ts + \mu n) u^\mu u^\nu] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0 \]

\[ \partial_\mu (nu^\mu) + \partial_\mu \nu^\mu = 0 \]
Dissipative terms

Standard textbook manipulations (single U(1) charge)

$$-\frac{u_\nu}{T} \times \partial_\mu \left[ (Ts + \mu n)u^\mu u^\nu \right] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0$$

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\[\partial_\mu (su^\mu) = \quad \frac{\mu}{T} \partial_\mu \nu^\mu + \frac{1}{T} \quad u_\nu \partial_\mu \tau^{\mu\nu}\]
Dissipative terms

Standard textbook manipulations (single U(1) charge)

\[-\frac{u_\nu}{T} \times \partial_\mu \left[ \left( Ts + \mu n \right) u^\mu u^\nu \right] + \partial^\nu P + \partial_\mu \tau^{\mu\nu} = 0\]

\[+\]

\[-\frac{\mu}{T} \times \partial_\mu (n u^\mu) + \partial_\mu \nu^\mu = 0\]

\[\partial_\mu (s u^\mu - \frac{\mu}{T} \nu^\mu) = \frac{\mu}{T} \partial_\mu \nu^\mu + \frac{1}{T} u_\nu \partial_\mu \tau^{\mu\nu}\]
Dissipative terms

Standard textbook manipulations (single U(1) charge)

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\[\partial_\mu (s u^\mu - \frac{\mu}{T} \nu^\mu) = -\partial_\mu \frac{\mu}{T} \quad \nu^\mu - \frac{1}{T} \partial_\mu u_\nu \quad \tau^{\mu \nu}\]
Dissipative terms

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entropy current \( s^\mu \)
Dissipative terms

Standard textbook manipulations (single U(1) charge)

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\[\partial_\mu (s u^\mu - \frac{\mu}{T} \nu^\mu) = -\partial_\mu \frac{\mu}{T} \nu^\mu - \frac{1}{T} \partial_\mu u_\nu \tau^{\mu \nu}\]

entropy current \( s^\mu \)

Positivity of entropy production constrains the dissipation terms: only three kinetic coefficients \( \eta, \zeta, \) and \( \sigma \) (right hand side positive-definite)
Mistake in holographic computations?

\[ S = \frac{1}{8\pi G} \int d^5 x \sqrt{-g} \left( R - 12 - \frac{1}{4} F_{AB}^2 + \frac{4\kappa}{3} \epsilon^{LABCD} A_L F_{AB} F_{CD} \right) \]

\[ \Box A^\mu \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta} \]

\[ 0 \quad r \quad j \quad k \]
Mistake in holographic computations?

\[ S = \frac{1}{8\pi G} \int d^5 x \sqrt{-g} \left( R - 12 - \frac{1}{4} F_{AB}^2 + \frac{4\kappa}{3} \epsilon^{LABCD} A_L F_{AB} F_{CD} \right) \]

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\[ A_i \sim u_i \]
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\[ \Box A^\mu \sim \epsilon^{\mu\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta} \quad A_i \sim u_i \]

- This lead to correction to the gauge field

- \( \delta A_i \sim \epsilon_{ijk} \partial_j u_k \)

- Current is read out from asymptotics of A near the boundary: \( j \sim \omega \)
Resolution

- DTS, Surówka: by carefully following Landau-Lifshitz argument of positive entropy production
- CVE is not only allowed, but is required if anomaly is present

\[ j^a = \cdots + \xi^a (\nabla \times v) \]

\[ \xi^a = C^{abc} \mu^b \mu^c + \cdots \]

This formula reproduces the value of the coefficient in N=4 SYM theory
CVE at finite $T$

- What we missed: if the current is $U(1)$ axial type, there is a possible extra contribution to the CVE coefficient

$$\xi = C\mu^2 + C'T^2$$

$C'$ was found to be related to gravitational contribution to triangle anomaly ("gravitational anomaly") both at weak and strong coupling Landsteiner, Megías, Pena-Benitez

The connection with "gravitational anomaly" now seems to be well understood Jensen Loganayagam Yarom
Weak coupling

QFT → Kinetic theory → Hydrodynamics

Strong coupling
(no kinetic theory)

Example of kinetic theory: Landau’s Fermi liquid theory
Weak coupling

anomalies

QFT → Kinetic theory → hydrodynamics

strong coupling (no kinetic theory)

Example of kinetic theory: Landau’s Fermi liquid theory
Weak coupling

- QFT
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Strong coupling (no kinetic theory)

Example of kinetic theory: Landau’s Fermi liquid theory
Weak coupling

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anomalies → anomalies? → anomalies

Example of kinetic theory: Landau’s Fermi liquid theory

strong coupling
(no kinetic theory)
Can kinetic theory reproduce anomalies?

- In kinetic theory one follows the evolutions of particles in phase space
- By definition the particle number is conserved
- How can one get nonconservation?
Berry curvature in momentum space

- Spin and orbital motions are locked
- Momentum change → Berry phase
- modifies the equation of motion
Semiclassical equation

\[ \dot{x} = \frac{\partial \epsilon_p}{\partial p} + \dot{p} \times \Omega \]

\[ \dot{p} = E + \dot{x} \times B \]

\[ \Omega = \pm \frac{p}{2|p|^3} \]

Magnetic monopole in momentum space

Berry curvature

Berry phase

M. Stephanov: Lorentz invariance

Chang, Niu
Hamiltonian interpretation

\[ \dot{\xi^a} = \{H, \xi^a\} \quad \{\xi^a, \xi^b\} = \omega^{ab} \]

\[ \{p_i, p_j\} = -\frac{\epsilon_{ijk}B_k}{1 + \mathbf{B} \cdot \Omega} \quad \{x_i, x_j\} = \frac{\epsilon_{ijk}\Omega_k}{1 + \mathbf{B} \cdot \Omega} \]

\[ \{p_i, x_j\} = \frac{\delta_{ij} + \Omega_i B_j}{1 + \mathbf{B} \cdot \Omega} \]
Anomalies from Berry curvature

• Solving the equation of motion for a single particle

\[
\dot{x} = (1 + \Omega \cdot B)^{-1} [v + \mathbf{E} \times \Omega + (\Omega \cdot v)\mathbf{B}]
\]

\[
\dot{p} = (1 + \Omega \cdot B)^{-1} [\mathbf{E} + v \times \mathbf{B} + (\mathbf{E} \cdot \mathbf{B})\Omega]
\]

from this one derive the Liouville equation

\[
(1 + \Omega \cdot B) \frac{\partial n_p}{\partial t} + \cdots + (\mathbf{E} \cdot \mathbf{B}) \left( \Omega \cdot \frac{\partial n_p}{\partial p} \right) = 0
\]
Anomaly from kinetic theory

\[ n(t, x) = \int \frac{d\mathbf{p}}{(2\pi)^3} \left( 1 + \mathbf{\Omega} \cdot \mathbf{B} \right) n_p(t, x) \]

\[ \frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = -(\mathbf{E} \cdot \mathbf{B}) \int \frac{d\mathbf{p}}{(2\pi)^3} \mathbf{\Omega} \cdot \frac{\partial n_p}{\partial \mathbf{p}} \]

\[ \frac{\partial n_p}{\partial \mathbf{p}} \\
\]

flux of \( \Omega \) through the Fermi sphere

\[ \partial_\mu j^\mu = \frac{1}{4\pi^2} \mathbf{E} \cdot \mathbf{B} \]

anomaly is reproduced
Weak coupling

QFT → Kinetic theory → hydrodynamics

strong coupling
(no kinetic theory)
Weak coupling

QFT $\rightarrow$ Kinetic theory $\rightarrow$ hydrodynamics

anomalies

strong coupling
(no kinetic theory)
Weak coupling

anomalies

QFT \[\rightarrow\] Kinetic theory \[\rightarrow\] hydrodynamics

strong coupling (no kinetic theory)

anomalies
Weak coupling

QFT → Kinetic theory → hydrodynamics

anomalies

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anomalies

anomalies
Conclusion

- Effects of anomalies in hydrodynamics
- Nontrivial use of gauge/gravity duality
- Kinetic theory knows about anomalies through the Berry curvature
- Deep connection between quantum and classical physics
Single-quasiparticle physics

\[ S = \int dt \left( p^i \dot{x}^i - \epsilon(p) + A_0 + A_i \dot{x}^i - A_i(p) \dot{p}^i \right) = \int dt \left( -\omega_a \dot{\xi}^a - H(\xi) \right) \]

equation of motion: \[ \omega_{ab} \dot{\xi}^b + \partial_a H = 0 \quad \omega_{ab} = \partial_a \omega_b - \partial_b \omega_a \]

Hamiltonian interpretation:

\[ \dot{\xi}^a = \{H, \xi^a\} \quad \{\xi^a, \xi^b\} = \omega^{ab} \]

\[ \{p_i, p_j\} = -\frac{\epsilon_{ijk} B_k}{1 + \mathbf{B} \cdot \Omega} \quad \{x_i, x_j\} = \frac{\epsilon_{ijk} \Omega_k}{1 + \mathbf{B} \cdot \Omega} \]

\[ \{p_i, x_j\} = \frac{\delta_{ij} + \Omega_i B_j}{1 + \mathbf{B} \cdot \Omega} \]
Example: \( B=0 \)

\[
\{p_i, p_j\} = 0 \quad \{p_i, x_j\} = \delta_{ij} \quad \{x_i, x_j\} = \epsilon_{ijk} \Omega_k
\]

\[
L = x \times p \quad \{L_i, L_j\} \neq -\epsilon_{ijk} L_k
\]

\[
J = L \pm \frac{p}{2|p|} \quad \{J_i, J_j\} = -\epsilon_{ijk} J_k
\]

spin