Chiral Shock Waves

Srimoyee Sen,
University of Arizona

In collaboration with Naoki Yamamoto (Keio University)

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Outline

- Chiral transport phenomenon.
- What is a shock wave?
- Shock wave in nonchiral matter.
- Shockwave in chiral matter.
- Example and implications.
Ideal and dissipative hydrodynamics

- Energy momentum conservation and particle number conservation:

\[ \partial_\mu T^{\mu\nu} = 0, \]
\[ \partial_\mu j^\mu = 0. \]

Where

\[ T^{\mu\nu} = hu^\mu u^\nu - p g^{\mu\nu}, \quad j^\mu = nu^\mu, \]
\[ h = \epsilon + p, \quad u^\mu = \gamma(1, v). \]

- Dissipative processes require additional terms in the conserved quantities – to be constrained by the second law of thermodynamics.
Consider massless fermions of single chirality.

In the presence of a background electromagnetic field

\[ \partial_\mu T^{\mu \nu} = F^{\nu \lambda} j_\lambda, \quad \partial_\mu j^\mu = C E^\mu B_\mu \]

External fields add further dissipative components to conserved currents where as anomaly adds nondissipative terms.
Chiral Transport

- Ignoring dissipation the constitutive relations become in Landau frame:

\[ j^\mu = n u^\mu + \xi \omega^\mu + \xi_B B^\mu \]

\[ \xi = \frac{C}{2} \mu^2 \left( 1 - \frac{2}{3} \frac{n\mu}{\epsilon + P} \right) + \frac{D}{2} T^2 \left( 1 - \frac{2n\mu}{\epsilon + P} \right) \]

\[ \xi_B = C \mu \left( 1 - \frac{1}{2} \frac{n\mu}{\epsilon + P} \right) - \frac{D}{2} \frac{nT^2}{\epsilon + P} \]

\[ \omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho \]

\[ B^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta} \]

Fluid flow (sonic)

- Effects of compressibility for high velocity fluid flow important.

- Two types: subsonic and supersonic flow.

- Supersonic flow of two types:
  - Steady continuous flow.
  - Surface of discontinuity $\rightarrow$ Shock waves.
• Velocity, pressure, density and temperature can be discontinuous across a surface perpendicular to the flow.

• Certain boundary conditions must be satisfied at this surface.

• Quantities like mass flux and energy flux should remain continuous across the surface for non-relativistic and relativistic systems respectively.
Nonchiral relativistic Shock Waves

• Shockwave traveling along 'x' axis.

• The region behind and ahead of the shock wave front denoted by 1 and 2.

• Impose continuity in particle number flux, energy flux etc.

\[
\begin{align*}
\dot{j}_1^x &= \dot{j}_2^x \\
T_1^{xx} &= T_2^{xx} \\
T_1^{0x} &= T_2^{0x} \\
T_1^{yx} &= T_2^{yx} \\
T_1^{zx} &= T_2^{zx}
\end{align*}
\]
Pressure-volume relation

\[ v_1 = \sqrt{\frac{(p_2 - p_1)(\epsilon_2 + p_1)}{(\epsilon_2 - \epsilon_1)(\epsilon_1 + p_2)}} \]

\[ v_2 = \sqrt{\frac{(p_2 - p_1)(\epsilon_1 + p_2)}{(\epsilon_2 - \epsilon_1)(\epsilon_2 + p_1)}} \]

The adiabatic:

\[ \frac{v_1}{V_1 \sqrt{1 - v_1^2}} - \frac{v_2}{V_2 \sqrt{1 - v_2^2}} = 0 \]

\[ h_1^2 V_1^2 - h_2^2 V_2^2 + (p_2 - p_1)(h_1 V_1^2 + h_2 V_2^2) = 0 \]
Weak shock waves

\[ \epsilon_2 \to \epsilon_1 \quad p_2 \to p_1 \quad \Delta \epsilon \equiv \epsilon_2 - \epsilon_1 \quad \Delta p \equiv p_2 - p_1 \]

\[ v_1^2 = (c_{s1})^2 \left( 1 + \frac{\Delta \epsilon}{\epsilon_1} (1 - c_{s1}^2) + .. \right) \]

\[ v_2^2 = (c_{s2})^2 \left( 1 - \frac{\Delta \epsilon}{\epsilon_1} (1 - c_{s2}^2) + .. \right) \]

\[ c_s^2 = \frac{dp}{d\epsilon} \]
Compression and rarefaction shock waves

\[ \epsilon_2 > \epsilon_1 \quad \implies \quad (v_1)^2 > (v_2)^2 \]

\[ p_2 > p_1 \quad \text{follows from:} \quad c_s^2 = \lim_{2 \to 1} \frac{p_2 - p_1}{\epsilon_2 - \epsilon_1} > 0 \]

\[ \implies \text{compression} \]

\[ \epsilon_2 < \epsilon_1 \quad \implies \quad (v_1)^2 < (v_2)^2 \]

\[ p_1 > p_2 \quad \text{follows from:} \quad c_s^2 = \lim_{2 \to 1} \frac{p_2 - p_1}{\epsilon_2 - \epsilon_1} > 0 \]

\[ \implies \text{rarefaction} \]
Entropy discontinuity

\[ \Delta H = H_2 - H_1 \quad \Delta S = S_2 - S_1 \]

\[ \Delta V = V_2 - V_1 \]

Define

Expand the pressure-volume relation using:

\[ \Delta H = T \Delta S + V_1 \Delta p + \frac{1}{2} \frac{\partial V}{\partial p} \Delta p^2 + \frac{1}{6} \frac{\partial^2 V}{\partial p^2} \Delta p^3 + \cdots \]

\[ \Delta V = \left. \frac{\partial V}{\partial p} \right|_1 \Delta p + \frac{1}{2} \left. \frac{\partial^2 V}{\partial p^2} \right|_1 (\Delta p)^2 + \frac{1}{6} \left. \frac{\partial^3 V}{\partial p^3} \right|_1 (\Delta p)^3 \]

\[ + \left. \frac{\partial V}{\partial S} \right|_1 \Delta S + \cdots \]
Entropy discontinuity and shockwaves

\[
\Delta S = \frac{1}{12H_1T} \left. \frac{\partial^2 (HV)}{\partial p^2} \right|_1 (\Delta p)^3 + O((\Delta p)^4)
\]

For any realistic equation of state: \( \frac{\partial^2 (HV)}{\partial p^2} > 0 \)

Second law of thermodynamics:

\[
S_2 > S_1 \quad \Rightarrow \quad p_2 > p_1 \quad \checkmark
\]

\[
p_1 > p_2 \quad \times
\]

Hence, only compression shock waves allowed!
Chiral shock waves

- Does the pressure entropy discontinuity depend on chiral transport? How?

- To answer choose a particular limit.

- Set $B^\mu$ to 0 such that, but $\omega^\mu$ is finite

$$j^\mu = n\omega^\mu + \xi \omega^\mu$$

- Fermions with chemical potential $\mu$ and temperature $T$ such that $T/\mu \ll 1$
Chiral shock waves

- Shock wave front traveling along 'x'.
- Vorticity chosen to be along 'x',
  \[ \omega_x = \omega, \omega_y = \omega_z = 0 \]
- Hydrodynamics makes sense when \( \omega \ll \mu \)
- Back of the wave-front denoted by '1', Front denoted by '2'.
Chiral shock waves

• Due to this vorticity, we cannot go to a frame with $v_1^y = v_2^y = v_1^z = v_2^z = 0$ everywhere.

• Consider the regime $|\omega|\rho \ll 1$ where the distance from axis:

\[ \rho = \sqrt{y^2 + z^2} \]

• In that case the speed perpendicular to the direction of vorticity

\[ v_\perp = \omega \rho (1 - v_x^2) + O((\omega \rho)^2) \]
• From the continuity equation we have to have
\[ v_1^y = v_2^y \quad \text{and} \quad v_1^z = v_2^z \quad \text{or} \quad v_1^\perp = v_2^\perp \]

• This implies
\[ \omega_1 \left( 1 - (v_1^x)^2 \right) = \omega_2 \left( 1 - (v_2^x)^2 \right) \]

• In the limit \( |\omega| \rho \ll 1 \) the expressions for \( v_1^2 \) and \( v_2^2 \) are given by their nonchiral version.
We know the expansion of the LHS in terms of $\Delta S$ and $\Delta p$.

The RHS is a function of $\mu_1, \mu_2, T_1, T_2$ as well.

We do not know the expansion of $\Delta \mu \equiv \mu_2 - \mu_1$ and $\Delta T \equiv T_2 - T_1$ in terms of $\Delta p$ and $\Delta S$. 

\[
\frac{v_1}{V_1 \sqrt{1 - v_1^2}} - \frac{v_2}{V_2 \sqrt{1 - v_2^2}} = -(\xi_1 \omega_1 - \xi_2 \omega_2)
\]
At this point we need to express $\mu$ and $T$ as a function of $p$ and $S$.

Assume noninteracting Fermi gas to do so:

\[
\begin{align*}
n &= \frac{\mu^3}{6\pi^2} + \frac{\mu T^2}{6}, \\
p &= \frac{\epsilon}{3} = \frac{\mu^4}{24\pi^2} + \frac{\mu^2 T^2}{12}, \\
S &= \frac{\pi^2 T}{\mu}.
\end{align*}
\]
Entropy discontinuity

The pressure volume relation expanded:

\[ \Delta S \approx \frac{216\pi^6}{\mu_1^{11} T_1} (\Delta p)^3 - \frac{\omega_1 \lambda}{T_1} \frac{36\sqrt{2\pi^4}}{\mu_1^8} (\Delta p)^2 + .. \]

Dominates for \[ \omega_1 < \frac{\Delta p}{\mu_1^3} 3\sqrt{2\pi^2} \]

And we are back to nonchiral shockwaves..
Entropy discontinuity

For \( \omega_1 > \frac{\Delta p}{\mu_1^3} 3 \sqrt{2\pi^2} \) dominates

\[
\Delta S \approx \frac{216 \pi^6}{\mu_1^{11} T_1} (\Delta p)^3 - \frac{\omega_1 \lambda}{T_1} \frac{36 \sqrt{2\pi^4}}{\mu_1^8} (\Delta p)^2 + ..
\]

And the entropy discontinuity:

\[
\Delta S \approx -\frac{\omega_1 \lambda}{T_1} \frac{36 \sqrt{2\pi^4}}{\mu_1^8} (\Delta p)^2 + ..
\]
Entropy discontinuity

- $\Delta S$ is quadratic in $(\Delta p)$ in chiral matter instead of being cubic as in nonchiral matter.

- Both rarefaction and compression shockwaves are allowed in chiral matter provided chiral transport dominates!

- Depending on the chirality of fermion, the wave can only propagate either along the vorticity or opposite to the vorticity, but not both.
Conclusion

• We find that rarefaction shockwaves are allowed by the second law of thermodynamics in chiral matter.

• Our result is exemplified in a limit \( T/\mu \ll 1 \) in a vorticity.

• We expect the qualitative form to hold in other regimes such as that of high temperature and nonzero magnetic field as well.