Derivation of Covariant Chiral Kinetic Equation by Wigner Function

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Chirality, Vorticity and Magnetic Field in HIC, UCLA
Outline

- Wigner function in static-equilibrium conditions
- Derivation of CCKE in 4D in Wigner function approach
- Freedom of adding more terms to CCKE
- Coefficients of these terms are constrained by conservation laws and other relations
- Chiral kinetic equation: from 4D to 3D
To describe dynamics of chiral fermions, we have to explicitly know their helicity (equivalently $p$), therefore we need to know information of $(t,x,p)$, that’s why we need kinetic theory in phase space.

- Classical kinetic approach: $f(t,x,p)$
- Quantum kinetic approach: $W(t,x,p)$
Wigner Function in 4D

- Gauge invariant Wigner operator/function in background EM fields

\[ W(x, p) = \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \left\langle \bar{\psi}_\beta \left( x + \frac{1}{2} y \right) U \left( A, x + \frac{1}{2} y, x - \frac{1}{2} y \right) \psi_\alpha \left( x - \frac{1}{2} y \right) \right\rangle \]

Heinz, 1983
Vasak, Gyulassy, Elze, 1986, 1987

- Dirac equation in EM fields

\[ [i \gamma^\mu D_\mu(x) - m] \psi(x) = 0, \quad \bar{\psi}(x) \left[ i \gamma^\mu D^\dagger_\mu(x) - m \right] = 0 \]

- In homogeneous EM fields, quantum kinetic equation for Wigner function of massless fermions can be derived from Dirac Eq.

\[ \gamma_\mu \left( p^\mu + \frac{1}{2} i \nabla^\mu \right) W(x, p) = 0 \]

\[ \nabla^\mu \equiv \partial^\mu_x - QF^{\mu\nu} \partial^\nu_x \]

phase-space derivative
Wigner Function in 4D

- Wigner function decomposition in 16 generators of Clifford algebra

\[ W(x, p) = \frac{1}{4} \left[ \mathcal{F} + i \gamma_5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma_5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma_{\mu\nu} \mathcal{S}_{\mu\nu} \right] \]

  - scalar
  - p-scalar
  - vector
  - axial-vector
  - tensor

- Currents and energy-momentum tensor can be obtained from Wigner function components

\[ j^\mu = \int d^4 p \mathcal{V}_\mu , \quad j_5^\mu = \int d^4 p \mathcal{A}_\mu , \quad T^{\mu\nu} = \frac{1}{2} \int d^4 p p^{(\mu} \mathcal{V}^{\nu)} \]


Two sets of equations for WF components are derived from QKE for WF which are decoupled from each other. One set of equations is for the vector component $J^s_\mu(x, p)$ for chiral fermions (with chirality $s = \pm$),

$$p^\mu J^s_\mu(x, p) = 0$$
$$\nabla^\mu J^s_\mu(x, p) = 0$$
$$2s(p^\lambda J^\rho_\lambda J_s^\rho - p^\rho J^\lambda_\rho J_s^\lambda) = -\epsilon^{\mu\nu\lambda\rho}\nabla_\mu J^s_\nu$$

where $J^s_\mu(x, p)$ are defined as

$$J^s_\mu(x, p) = \frac{1}{2}[\gamma_\mu(x, p) + s\alpha_\mu(x, p)]$$
Perturbative solution

Perturbation in \((\partial^x_{\mu})^n\) and \((F_{\mu\nu})^n\). The solution at the 0-th and 1-st order

\[
\mathcal{J}_s^{(0)}(x, p) = p^\rho f_s \delta(p^2)
\]

\[
\mathcal{J}_s^{(1)}(x, p) = -\frac{s}{2} \tilde{\Omega}^\rho_\beta p_\beta \frac{df_s}{dp_0} \delta(p^2) - \frac{s}{p^2} Q \tilde{F}^\rho_\lambda p_\lambda f_s \delta(p^2)
\]

where \(f_s\) is the distribution function \((\mu_s = \mu + s\mu_5, p_0 = p \cdot u)\),

\[
f_s(x, p) = \frac{2}{(2\pi)^3} \left[ \Theta(p_0) f_{FD}(p_0 - \mu_s) + \Theta(-p_0) f_{FD}(-p_0 + \mu_s) \right]
\]

\[
f_{FD}(x) = \frac{1}{e^{\beta x} + 1}, \quad \text{Fermi-Dirac}
\]

\[\tilde{\Omega}^\rho_\beta = \frac{1}{2} \epsilon^\rho_\beta\mu\nu \Omega_{\mu\nu}\]

\[\tilde{F}^\rho_\lambda = \frac{1}{2} \epsilon^\rho_\lambda\mu\nu F_{\mu\nu}\]

\[\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu)\]
Static-equilibrium conditions

The Wigner function solution is obtained under static equilibrium conditions

\[
\Delta^{\sigma\alpha} \Delta^{\rho\beta} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \Delta^{\alpha\beta} \Delta^{\rho\sigma} \partial_\rho u_\sigma \right) = 0
\]

\[
T \Delta^{\sigma\rho} \partial_\rho \frac{\mu}{T} + QE^\sigma = 0
\]

\[
u^\rho \partial_\rho u^\sigma - \Delta^{\sigma\rho} \partial_\rho \ln T = 0
\]

\[
u^\sigma \partial_\sigma T + \frac{1}{3} T \Delta^{\rho\sigma} \partial_\rho u_\sigma = 0
\]

\[
\partial_\sigma \frac{\mu_5}{T} = 0, \quad u^\sigma \partial_\sigma \frac{\mu}{T} = 0
\]

Like Killing condition
Becattini (2012)
Becattini, Bucciantini, Grossi, Tinti (2015)
Becattini, Grossi (2015)
Simplified conditions for constant temperature

With constant temperature, the above conditions are reduced to following simplified form

\[ \partial_\sigma \mu_5 = 0, \quad \partial_\sigma T = 0 \]
\[ \partial^\rho u^\sigma + \partial^\sigma u^\rho = 0 \]
\[ \partial_\sigma \mu = -QE_\sigma \]

Like Killing condition
Becattini (2012)
Becattini, Bucciantini, Grossi, Tinti (2015)
Becattini, Grossi (2015)
We insert the Wigner function solutions into

$$\nabla_{\mu}[\mathcal{I}_{(0)s} + \mathcal{I}_{(1)s}] = 0$$

The zero-th order is evaluated as

$$\nabla_{\mu}[p^{\mu}f_{s}\delta(p^2)] = (\partial^{\mu}_{\chi} - QF^{\mu\nu}\partial_{\nu}^{\mu})[p^{\mu}f_{s}\delta(p^2)]$$

$$= \delta(p^2)p^{\mu}\nabla_{\mu}f_{s}$$

where we used $p^{\mu}\nabla_{\mu}\delta(p^2) = 0$ and $\nabla_{\mu}p^{\mu} = 0$. The first order can also be evaluated similarly.
We combine the 0-th and 1st order contribution we obtain

\[ \nabla_\mu [J^{(0)}_\mu + J^{(1)}_\mu] = \delta(p^2) \left[ p_\mu \nabla_\mu f_s + sQ \frac{1}{p^2} \Omega^\mu_\lambda p_\lambda \tilde{F}_{\mu\kappa} p_\kappa f'_s \right. \\
- \left. sQ \frac{1}{p^2} \tilde{F}^\mu_\lambda p_\lambda (\nabla_\mu f'_s) - sQ \frac{1}{p^2} \tilde{F}^\mu_\lambda p_\lambda (\nabla_\mu f_s) \right] = 0 \]
Derivation of CCKE

Further simplification gives

\[
\nabla^\mu \equiv \partial^\mu - QF^{\mu\nu} \partial^\nu
\]

\[
\nabla_\mu f_s
\]

\[
\left[ \left( p^\mu - sQ \frac{1}{p^2} \tilde{F}^{\mu\lambda} p_\lambda - s \frac{p_0}{p^2} \tilde{\Omega}^{\mu\lambda} p_\lambda + \frac{s}{2} \tilde{\Omega}^{\mu\lambda} u_\lambda \right) \nabla_\mu f_s + \left( -s \frac{1}{2} \tilde{\Omega}^{\mu\lambda} p_\lambda \Omega_{\mu\nu} + sQ \frac{1}{p^2} \Omega^{\mu\lambda} p_\lambda \tilde{F}_{\mu\kappa} p_\kappa \Omega_{\nu} \right) \partial^\nu f_s \right] \delta(p^2) = 0,
\]

which can be cast into the form

\[
\delta(p^2) \left( \frac{dx^\mu}{d\tau} \partial^\mu f_s + \frac{dp^\mu}{d\tau} \partial^\mu f_s \right) = 0
\]
Here $dx^\mu / d\tau$ and $dp^\mu / d\tau$ are given by

$$m_0 \frac{dx^\mu}{d\tau} = p^\mu - sQ \frac{1}{p^2} \tilde{F}^\mu_\lambda p_\lambda + s \left( \frac{1}{2} - \frac{p_0^2}{p^2} \right) \omega^\mu + s \frac{p_0}{p^2} (p \cdot \omega) u^\mu + X^\mu$$

$$m_0 \frac{dp^\mu}{d\tau} = QF^{\mu_\nu} p_\nu + sQ^2 \frac{p^\mu}{4p^2} F^{\nu_\lambda} \tilde{F}_{\nu_\lambda}$$

$$+ \frac{1}{2} sQ (E \cdot \omega) u^\mu - sQ \frac{1}{p^2} (p \cdot \omega) (p \cdot E) u^\mu + sQ \frac{1}{p^2} p_0 (p \cdot \omega) E^\mu + Y^\mu$$
The new terms $X^\mu$ and $Y^\mu$ can be added

\[
\frac{dx^\mu}{d\tau} \leftarrow X^\mu = sC_1(p, u)\omega^\mu + sC_2(p, u)(p \cdot \omega)u^\mu
+ sC_3(p, u)(p \cdot \omega)\bar{p}^\mu
\]

\[
\frac{dp^\mu}{d\tau} \leftarrow Y^\mu = -sQ[C_1(p, u)(\omega \cdot E) + C_3(p, u)(p \cdot \omega)(p \cdot E)]u^\sigma
+ sQ C_4(p, \omega)\bar{p}^\sigma
\]

which satisfy the equation

\[
X^\sigma \partial^x_{\sigma} f_s + Y^\sigma \partial^p_{\sigma} f_s = 0
\]
We assume following forms for these unknown functions

\[
C_1(p, u) = C_{10} + C_{11} \frac{p_0^2}{p^2}
\]

\[
C_2(p, u) = C_{20} \frac{p_0}{p^2} + C_{21} \frac{1}{p_0}
\]

\[
C_3(p, u) = C_{30} \frac{1}{p^2}
\]

\[
C_4(p, \omega) = C_{40}(\omega \cdot E) \frac{1}{p_0} + C_{41} \frac{1}{p^2 p_0} (p \cdot \omega)(p \cdot E)
\]

where \( \{C_{10}, C_{11}, C_{20}, C_{21}, C_{30}, C_{40}, C_{41}\} \) are dimensionless constants to be determined.
The currents for chiral fermions with chirality \( s = \pm 1 \) are given by,

\[
j_{s}^{\mu} = \int d^{4}p \frac{\delta(p^{2})}{m_{0}} \frac{d\chi^{\mu}}{d\tau} f_{s} = j_{s}^{\mu}(EM) + j_{s}^{\mu}(\omega)
\]

where for the current from EM field

\[
j_{s}^{\mu}(EM) = -sQ \int d^{4}p \frac{1}{p^{2}} \tilde{F}^{\mu\lambda} p_{\lambda} f_{s} = \xi_{B}^{s} B^{\mu}
\]

\[
j^{\mu}(EM) = (\xi_{B}^{+} + \xi_{B}^{-}) B^{\mu} = \frac{Q}{2\pi^{2}} \mu_{5} B^{\mu}
\]

\[
j_{5}^{\mu}(EM) = (\xi_{B}^{+} - \xi_{B}^{-}) B^{\mu} = \frac{Q}{2\pi^{2}} \mu B^{\mu}
\]
Constraints from currents

For the current from vorticity

\[ j^\mu_s(\omega) = \left( C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} + 1 \right) \xi_s \omega^\mu \]

\[ j^\mu(\omega) = j^\mu_+(\omega) + j^\mu_-(\omega) \rightarrow \frac{1}{\pi^2} \mu \mu_5 \omega^\mu \]

\[ j^\mu_5(\omega) = j^\mu_+(\omega) - j^\mu_-(\omega) \]

\[ \rightarrow \left[ \frac{1}{6} T^2 + \frac{1}{2\pi^2} (\mu^2 + \mu_5^2) \right] \omega^\mu \]

which gives the constraints

\[ C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} = 0 \]
The energy momentum tensor in the relativistic chiral kinetic theory can be obtained by

\[
T^{\rho \sigma} = \frac{1}{2} m_0 \int d^4 p \delta(p^2) \sum_s p(\rho) \frac{d x^{\rho}}{d \tau} f_s
\]

\[
= T^{\rho \sigma} (\text{EM}) + T^{\rho \sigma} (\omega)
\]

\[
= \frac{1}{2} Q \xi u^{(\rho B^{\sigma})} + \left( \frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + \frac{1}{4} C_{20} - \frac{1}{6} C_{21} + \frac{3}{4} \right) n_5 u^{(\rho \omega^{\sigma})}
\]

\[
= \frac{1}{2} Q \xi u^{(\rho B^{\sigma})} + n_5 u^{(\rho \omega^{\sigma})}
\]

which gives the constraints

\[
\frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + \frac{1}{4} C_{20} - \frac{1}{6} C_{21} + \frac{3}{4} = 1
\]
We combine two constraints from conservation laws

\[ C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} = 0 \]
\[ C_{20} - \frac{2}{3} C_{21} = 1 \]

We see that \( C_{20} \) and \( C_{21} \) cannot all be zero.
Chiral kinetic equation: from 4D to 3D

With CCKE in 4D, we can obtain its 3D version by integrating over $p_0$,

$$I = \int dp_0 \delta(p^2) \left[ \frac{dx^\sigma}{d\tau} \partial^x_s + \frac{dp^\rho}{d\tau} \partial^p_s \right]$$

$$= I_{x0} + I_x + I_{p0} + I_p$$

Each term has three contributions

$$l_j = l_j(0) + l_j(EM) + l_j(\omega)$$

where $j = x0, p0, x, p$. 
For an on-shell particle the energy is not an independent phase space variable, its rate \( \frac{dE_p}{d\tau} \) from \( I_{p0} \) can be determined by

\[
\frac{dE_p}{d\tau} = \frac{p}{|p|} \cdot \frac{dp}{d\tau}
\]

So in derivation of the 3D chiral kinetic equation from the 4D one, the \( p_0 \) degree of freedom is fixed and is not a kinematic variable in the 3D kinetic equation. We can extract the EM field contribution to \( \frac{dp}{d\tau} \) from \( I_p(\text{EM}) \)

\[
\frac{dp}{d\tau}(\text{EM}) = Q \left( \mathbf{E} + \frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{B} \right) + sQ^2(\mathbf{E} \cdot \mathbf{B})p \frac{1}{2|\mathbf{p}|^3}
\]
The energy rate from the vorticity can be obtained from $I_{p_0}(\omega)$,

$$\frac{dE_p}{d\tau}(\omega) = -sQ(C_{30} + 1) \frac{1}{2|p|}(E \cdot \omega)$$

$$+ sQ(C_{30} + 1) \frac{3}{2|p|^3}(p \cdot \omega)(p \cdot E)$$

We can compare it with $dp/d\tau$ extracted from $I_p(\omega)$,

$$\frac{dp}{d\tau}(\omega) = sQ \left[ \frac{1}{|p|^2}(p \cdot \omega)E - C_{40} \frac{1}{|p|^2}(\omega \cdot E)p \right.$$

$$\left. - C_{41} \frac{2}{|p|^4}(p \cdot \omega)(p \cdot E)p \right]$$
Enforcing $\frac{dE_p}{d\tau} = \frac{p}{|p|} \cdot \frac{dp}{d\tau}$, the coefficients must satisfy

\begin{align*}
C_{40} &= \frac{1}{2} (C_{30} + 1) \\
C_{41} &= \frac{1}{2} - \frac{3}{4} (C_{30} + 1)
\end{align*}

So in this case we obtain $dp/d\tau$ from vorticity as

\[
\frac{dp}{d\tau}(\omega) = sQ \left[ \frac{1}{|p|^2} (p \cdot \omega) E - (C_{30} + 1) \frac{1}{2|p|^2} (\omega \cdot E)p \\
+ (1 + 3C_{30}) \frac{1}{2|p|^4} (p \cdot \omega)(p \cdot E)p \right]
\]
From from $I_{x0}$ and $I_x$ we obtain

\[
\frac{dx_0}{d\tau} = 1 + \mathcal{C}_B sQ(\Omega \cdot B) + \left(4 - \frac{2}{3} C_{21}\right) s|\mathbf{p}|(\Omega \cdot \omega),
\]

\[
\frac{dx}{d\tau} = \hat{p} + sQB(\hat{p} \cdot \Omega) + \mathcal{C}_E sQ(E \times \Omega)
\]

\[+ s \left(1 - \frac{1}{2} C_{30}\right) \frac{\omega}{|\mathbf{p}|} + 3sC_{30}(\Omega \cdot \omega)\mathbf{p},\]

where where we have used the constraints about $C_{10}$, $C_{11}$, $C_{20}, C_{21}$ and $C_{30}$.
CKE in 3D

[Son,Yamamoto 2012,2013; Stephanov,Yin 2012; Chen,Pu,Wang,Wang 2012; Chen,Son,Stephanov,Yee,Yin 2014; Kharzeev,Stephanov,Yee 2016; Hidaka,Pu,Yang 2016; Mueller,Venugopalan 2017]

\[
\begin{align*}
\frac{dx_0}{d\tau} &= 1 + \mathcal{C}_B s Q (\Omega \cdot B) + \left(4 - \frac{2}{3} C_{21}\right) s |p| (\Omega \cdot \omega) \\
\frac{dx}{d\tau} &= \hat{p} + s Q B (\hat{p} \cdot \Omega) + \mathcal{C}_E s Q (E \times \Omega) \\
&\quad + \left(1 - \frac{1}{2} C_{30}\right) s \frac{\omega}{|p|} + 3 C_{30} s (\Omega \cdot \omega)p \\
\frac{dp}{d\tau} &= Q \left( E + \frac{p}{|p|} \times B \right) + s Q^2 (E \cdot B)p \frac{1}{2|p|^3} \\
&\quad + s Q \frac{1}{|p|^2} \left[ (p \cdot \omega)E - \frac{1}{2} (C_{30} + 1)(\omega \cdot E)p \\
&\quad + \frac{1}{2} (1 + 3 C_{30}) \frac{1}{|p|^2} (p \cdot \omega)(p \cdot E)p \right]
\end{align*}
\]
Coefficients in 3D-CKE

- CKE in 3D is not uniquely determined due to free coefficients \( \{C_{21}, C_{30}, C_B, C_E\} \)
- \( C_B, C_E = 1 \) or 2. The freedom to choose \( C_B \) and \( C_E \) is because the integration over \( \mathbf{p} \) of their corresponding terms are all vanishing, we can make choices as to keep or drop them following some physical reasonings. The \( C_E = 1 \) is consistent to the previous result.
- With the coefficients \( \{C_{21}, C_{30}, C_B, C_E\} = \{0, 0, 1, 1\} \) we reproduce our previous result of [Chen, Pu, Wang, Wang, PRL 110, 262301(2013)].
Another possible choice of $C_{30}$ is $C_{30} = 2/3$. In this case the vorticity terms in $d\mathbf{x}/d\tau$ read

$$
\frac{d\mathbf{x}}{d\tau}(\omega) = \frac{s}{|\mathbf{p}|}(\hat{\mathbf{p}} \cdot \omega)\hat{\mathbf{p}} + \frac{s}{|\mathbf{p}|^2} 2\omega
$$

When calculating the vorticity contribution to the current by the integration over $\mathbf{p}$ for $d\mathbf{x}/d\tau$ times the distribution function, one can verify that the first term of contributes to $1/3$ of the chiral vortical effect while the second term contributes to the rest $2/3$. In comparison to the result of [Kharzeev,Stephanov,Yee, 1612.01674], the $1/3$ contribution corresponds to that from the spin-vorticity coupling energy, while the rest $2/3$ contribution corresponds to that from the magnetization current.
The CCKE is derived from 4D Wigner function by an improved perturbative method under the static equilibrium conditions. The CKE in 3D can be obtained by integration over the time component of the 4-momentum.

There is freedom to add more terms to the CCKE allowed by conservation laws.

In the derivation of the 3-dimensional equation, there is also freedom to choose coefficients of some terms in $d\mathbf{x}_0/d\tau$ and $d\mathbf{x}/d\tau$ whose 3-momentum integrals are vanishing.

So the 3-dimensional chiral kinetic equation derived from the CCKE is not uniquely determined in our current approach.

To go beyond the current approach, one needs a new way of building up the CKE in 3D from the CCKE or directly from the covariant equation for the Wigner function.